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AV-differential geometry: Euler–Lagrange equations[☆]

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Abstract

A general, consistent and complete framework for geometrical formulation of mechanical systems is proposed, based on certain structures on affine bundles (affgebroids) that generalize Lie algebras and Lie algebroids. This scheme covers and unifies various geometrical approaches to mechanics in the Lagrangian and Hamiltonian pictures, including time-dependent Lagrangians and Hamiltonians. In our approach, Lagrangians and Hamiltonians are, in general, sections of certain $\mathbb{R}$-principal bundles, and the solutions of analogs of Euler–Lagrange equations are curves in certain affine bundles. The correct geometrical and frame-independent description of Newtonian Mechanics is of this type.

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1. Introduction

In earlier papers [4,31] we developed a geometrical theory (AV-geometry) in which functions on a manifold are replaced by sections of $\mathbb{R}$-principal bundles (AV-bundles). It was used for frame-independent formulation of a number of physical theories. The general geometrical concepts and tools described in [4] were then applied to obtain the Hamiltonian picture in this affine setting.

In the present paper, in turn, we are concentrated on the Lagrangian picture and we derive analogs of the Euler–Lagrange equations in the AV-bundle setting. This is done completely geometrically and intrinsically, however with no reference to any variational calculus (which we plan to study in our forthcoming paper). We use the framework of what we call *special affgebroids* which is more general than that of *Lie affgebroids* used recently by other authors [9,14,15,18,19,25]. We get a larger class of Euler–Lagrange equations and larger class of possible models for physical theories which cover the ones considered in [9] as particular examples. The special affgebroids are related to Lie affgebroids in the way the general *algebroids* used in [5] are related to Lie algebroids. The other difference is that we do not use prolongations of Lie affgebroids that simplifies, in our opinion, the whole picture. This is possible, since

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in our method we do not follow the Klein's ideas [10] for geometric construction of Euler–Lagrange equations but rather some ideas due to Tulczyjew [27–29]. The dynamics obtained in this way is therefore implicit but we think it is the nature of the problem and this approach has the advantage that regularity of the Lagrangian plays no role in the main construction.

Note that the idea of using affine bundles for geometrically correct description of mechanical systems is not new and goes back to [26,28,30] (see also [20,21]).

The paper is organized as follows. In Section 2 we recall rudiments of the AV-geometry that will be needed in the sequel. Section 3 is devoted to the presentation of the concept of double affine bundle which is the fundamental concept in our approach. In Section 4 we recall the Lagrangian and Hamiltonian formalisms for general algebroids developed in [5]. On this fundamentals, special affgebroids as certain morphisms of double affine bundles and the corresponding Lagrangian and Hamiltonian formalisms are constructed and studied in Section 5. In particular, we derive analogs of Euler–Lagrange equations. We end up with some examples in Section 6.

2. Rudiments of the AV-geometry

We refer to [4] (see also [3,31]) for a development of the AV-geometry. Here we recall the basic notions very shortly, fixing notation and appropriate local coordinates.

In the standard differential geometry many constructions are based on the algebra $C^\infty(M)$ of smooth functions on the manifold M. In the geometry of affine values (AV-geometry in short) we replace $C^\infty(M)$ by the space of sections of certain affine bundle over M. Let $\zeta : \mathbf{Z} \to M$ be a one-dimensional affine bundle over the manifold M modelled on the trivial vector bundle $M \times \mathbb{R}$. Such a bundle will be called a *bundle of affine values* (*AV-bundle* in short). The difference of two sections of $\mathbf{Z}$ is then an ordinary function on the base. One can also say that AV-bundles are just one-dimensional *special affine bundles* $\mathbf{A} = (A, v_\mathbf{A})$, i.e. affine bundles A with a distinguished nowhere-vanishing section $v_\mathbf{A}$ of the model vector bundle $\mathsf{V}(A)$.

Every special affine bundle defines an AV-bundle $\mathsf{AV}(\mathbf{A})$ being the affine fibration $\mathbf{A} \to \underline{\mathbf{A}}$, where $\underline{\mathbf{A}} = \mathbf{A}/\langle v_\mathbf{A}\rangle$ with the free $\mathbb{R}$-action induced from translations in the direction of $-v_\mathbf{A}$.

The *phase bundle* $\mathsf{P}\mathbf{Z}$, defined analogously to T^*M, is an affine bundle modelled on T^*M and equipped with a canonical symplectic form. There is an *affine de Rham differential* $\mathrm{d} : \mathsf{Sec}(\mathbf{Z}) \to \mathsf{Sec}(\mathsf{P}\mathbf{Z})$. We write simply $\mathsf{P}\mathbf{A}$ for $\mathsf{P}(\mathsf{AV}(\mathbf{A}))$.

For a special affine bundle $\mathbf{A} = (A, v_\mathbf{A})$ over M its *special affine dual* $\mathbf{A}^\#$ is the affine subbundle in the vector bundle $A^\dagger$ of affine morphisms from fibers of A into $\mathbb{R}$, consisting of morphisms which are *special*, i.e. whose linear part maps $v_\mathbf{A}$ into 1. The constant map 1_A is the canonical distinguished nowhere-vanishing section of the model vector bundle. For example, $(A \times \mathbb{R})^\# = A^\dagger$. There is an obvious *special affine pairing*

$$\langle\cdot,\cdot\rangle_{sa} : \mathbf{A} \times_M \mathbf{A}^\# \to M \times \mathbb{R}, \qquad \langle a_m, \varphi_m\rangle_{sa} = (m, \varphi_m(a_m)), \tag{2.1}$$

which is a special affine morphism with respect to each argument, and a canonical identification $(\mathbf{A}^\#)^\# \simeq \mathbf{A}$.

A *Lie affgebroid* is an affine bundle A over M with a *Lie affgebra* bracket, i.e. an affine-linear map on the space of sections

$$[\cdot,\cdot] : \mathsf{Sec}(A) \times \mathsf{Sec}(\mathsf{V}(A)) \to \mathsf{Sec}(\mathsf{V}(A))$$

satisfying skew-symmetry and Jacobi identity, together with a morphism of affine bundles $\rho : A \to \mathsf{T}M$ (thus inducing a map from $\mathsf{Sec}(A)$ into vector fields on M) satisfying the anchor property. In other words, we assume the conditions

- skew-symmetry: $[a_1, a_2 - a_1] = [a_2, a_2 - a_1]$,
- Jacobi identity: $[a_1, [a_2, a_3 - a_2]] + [a_2, [a_3, a_1 - a_3]] + [a_3, [a_1, a_2 - a_1]] = 0$,
- anchor property: $[a_1, f(a_2 - a_1)] = f[a_1, a_2 - a_1] + \rho(a_1)(f)(a_2 - a_1)$

for all $a_i \in \mathsf{Sec}(A)$, $i = 1, \ldots, 3$, and for any smooth function f on M. A special *Lie affgebroid* is a Lie affgebroid on a special affine bundle such that the distinguished section is a central element of the corresponding bracket. Note that, due to skew-symmetry, one can also see the above Lie affgebra bracket as a bi-affine operation on sections of A if we put $[a_1, a_2] := [a_1, a_2 - a_1]$.

Example 2.1. Given a fibration $\xi\colon M \to \mathbb{R}$, take the affine subbundle $A \subset \mathsf{T}M$ characterized by $\xi_*(X) = \partial_t$ for $X \in \mathsf{Sec}(A)$. Then the standard bracket of vector fields in A, and $\rho\colon A \to \mathsf{T}M$ being just the inclusion, define on A a structure of a Lie affgebroid. This is the basic example of the concept of *affine Lie algebroid* developed in [19,25].

Given an AV-bundle $\mathbf{Z}$ over M, an *aff-Poisson bracket* on $\mathbf{Z}$ is a Lie affgebra bracket

$$\{\cdot,\cdot\}\colon \mathsf{Sec}(\mathbf{Z}) \times C^\infty(M) \to C^\infty(M) \tag{2.2}$$

such that

$$X_\sigma = \{\sigma,\cdot\}\colon C^\infty(M) \to C^\infty(M)$$

is a derivation represented by a vector field on M (called *the Hamiltonian vector field of* σ) for every $\sigma \in \mathsf{Sec}(\mathbf{Z})$.

For any AV-bundle $\mathbf{Z} = (Z, v_{\mathbf{Z}})$ the tangent bundle $\mathsf{T}\mathbf{Z}$ is equipped with the tangent $\mathbb{R}$-action. Dividing $\mathsf{T}\mathbf{Z}$ by the action we obtain the *Atiyah algebroid* of the principal $\mathbb{R}$-bundle $\mathbf{Z}$ which we denote by $\widetilde{\mathsf{T}}\mathbf{Z}$. It is a special Lie algebroid whose sections are interpreted as invariant vector field on the principal $\mathbb{R}$-bundle $\mathbf{Z}$. The distinguished section of $\widetilde{\mathsf{T}}\mathbf{Z}$ is represented by the fundamental vector field $\chi_{\mathbf{Z}}$ of the $\mathbb{R}$-action on $\mathbf{Z}$. The AV-bundle $\mathsf{AV}(\widetilde{\mathsf{T}}\mathbf{Z})$ is a bundle over $\mathsf{T}M$. The special affine dual for the special affine structure on $\widetilde{\mathsf{T}}\mathbf{Z}$ is $(\widetilde{\mathsf{T}}\mathbf{Z})^\# = \mathsf{P}\mathbf{Z}\times\mathbb{R}$. The special affine evaluation between $\mathsf{P}\mathbf{Z}\times\mathbb{R}$ and $\widetilde{\mathsf{T}}\mathbf{Z}$ comes from the interpretation of sections of $\mathsf{P}\mathbf{Z}$ as $\mathbb{R}$-invariant 1-forms α on $\mathbf{Z}$ such that $\langle\alpha,\chi_{\mathbf{Z}}\rangle = 1$ (i.e. principal connections on the $\mathbb{R}$-principal bundle $\mathbf{Z}$) that gives an affine-linear pairing $\langle\cdot,\cdot\rangle_\dagger : \mathsf{P}\mathbf{Z}\times_M \widetilde{\mathsf{T}}\mathbf{Z} \to \mathbb{R}$ and the identification $(\mathsf{P}\mathbf{Z})^\dagger = \widetilde{\mathsf{T}}\mathbf{Z}$ (cf. [3,4]). In this way sections of $(\mathsf{P}\mathbf{Z})^\dagger = \widetilde{\mathsf{T}}\mathbf{Z}$ represent *affine derivations* $D : \mathsf{Sec}(\mathbf{Z}) \to C^\infty(M)$ (i.e. such affine maps whose linear part $D_{\mathsf{v}} : C^\infty(M) \to C^\infty(M)$ is a derivation, thus a vector field on M). It is now obvious that affine biderivations $B : \mathsf{Sec}(\mathbf{Z}) \times \mathsf{Sec}(\mathbf{Z}) \to C^\infty(M)$ are sections of the bundle $\widetilde{\mathsf{T}}\mathbf{Z}\otimes_M \widetilde{\mathsf{T}}\mathbf{Z}$. In this picture, skew-symmetric affine biderivations are sections of $\wedge^2\widetilde{\mathsf{T}}\mathbf{Z}$ and they are uniquely determined by the brackets (2.2) defined by $\{a,v\} = B(a,a+v)$.

2.1. *Local affine coordinates and canonical identifications*

For a special affine bundle $\mathbf{A} = (A, v_{\mathbf{A}})$ of rank m, we use a section e_0 of the affine bundle A to identify A with its model special vector bundle $\mathsf{V}(\mathbf{A})$. The distinguished section $v_{\mathbf{A}}$ we can then extend to a basis $e_1,\dots,e_m$ of local sections of $\mathsf{V}(\mathbf{A})$ such that $e_m = v_{\mathbf{A}}$. In this way we can get a basis $e_0,\dots,e_m$ of local sections of the *vector hull* $\widehat{\mathbf{A}} = (A^\dagger)^*$ and the dual basis $e_0^*,\dots,e_m^*$ of local sections in $\widehat{\mathbf{A}}^* = \widehat{\mathbf{A}^\#}$ such that e_0^* represents the distinguished section of $\mathsf{V}(\mathbf{A}^\#)$. If we choose local coordinates (x^a) on the base manifold, these bases give rise to local coordinates $(x^a, y^0,\dots,y^m)$ and $(x^a,\xi_0,\dots,\xi_m)$ in the special vector bundles $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{A}}^*$, respectively, defined by

$$y^i = \iota_{e_i^*} \quad \text{for } i = 0,\dots,m-1, \qquad y^m = -\iota_{e_m^*}, \tag{2.3}$$

$$\xi_i = \iota_{e_i} \quad \text{for } i = 1,\dots,m, \qquad \xi_0 = -\iota_{e_0}, \tag{2.4}$$

where ι_e denotes the linear function on $\widehat{\mathbf{A}}^*$ corresponding to the section e of $\widehat{\mathbf{A}}$. With these coordinates the affine subbundles $\mathbf{A}$ and $\mathbf{A}^\#$ in $\widehat{\mathbf{A}}$ and $\widehat{\mathbf{A}^\#}$ are characterized by the equations $y^0 = -1$ and $\xi_m = -1$, respectively. Moreover $(x^a, y^1,\dots,y^m)$ and $(x^a,\xi_0,\dots,\xi_{m-1})$ are local coordinates in $\mathbf{A}$ and $\mathbf{A}^\#$, respectively, in which the special affine pairing $\langle\cdot,\cdot\rangle_{sa} : \mathbf{A}\times_M \mathbf{A}^\# \to \mathbb{R}$ reads

$$\langle (x,y^1,\dots,y^m),(x,\xi_0,\dots,\xi_{m-1})\rangle_{sa} = \sum_1^{m-1} y^i\xi_i - y^m - \xi_0. \tag{2.5}$$

Note that the base manifold $\underline{\mathbf{A}}$ of the AV-bundle $\zeta : \mathbf{A} \to \underline{\mathbf{A}}$ is an affine bundle $\underline{\eta} : \underline{\mathbf{A}} \to M$ with induced coordinates (x^a, y^i), $i = 1,\dots,m-1$, so $\zeta(x,y^1,\dots,y^m) = (x,y^1,\dots,y^{m-1})$ and $\chi_{\mathsf{AV}(\mathbf{A})} = -\partial_{y^m}$, i.e.

$$\mathbf{A} \ni (x,y^1,\dots,y^m) \mapsto ((x,y^1,\dots,y^{m-1}),y^m) \in \underline{\mathbf{A}}\times\mathbb{R}$$

represents a local isomorphism of $\mathsf{AV}(\mathbf{A})$ and $\underline{\mathbf{A}}\times\mathbb{R}$. Similar observations hold for $\mathbf{A}^\#$ and coordinates (x,ξ). Coordinates on special affine bundles used in the sequel will be always of this type.

In the space of sections of $\mathsf{AV}(\mathbf{A})$ one can distinguish *affine sections*, i.e. such sections

$$\sigma : \underline{\mathbf{A}} = A/\langle v_{\mathbf{A}}\rangle \to \mathbf{A}$$

which are affine morphisms. The space of affine sections will be denoted by $\mathsf{AffSec}(\mathsf{AV}(\mathbf{A}))$. We say that an operation $\mathsf{Sec}(\mathsf{AV}(\mathbf{A})) \times \mathsf{Sec}(\mathsf{AV}(\mathbf{A})) \to C^\infty(\underline{\mathbf{A}})$ is *affine-closed* if the product of any two affine sections is an affine function.

In the linear case there is a correspondence between Lie algebroid brackets on a vector bundle and linear Poisson structures on the dual bundle. In the affine setting we have an analog of this correspondence. Let X be a section of $\mathsf{V}(\mathbf{A})$. Since $\mathsf{V}(\mathbf{A})$ is a vector subbundle in $\widehat{\mathbf{A}}$, the section X corresponds to a linear function $\iota^\dagger_X$ on $\mathbf{A}^\dagger = (\widehat{\mathbf{A}})^*$. The function $\iota^\dagger_X$ is invariant with respect to the vertical lift of the distinguished section 1_A of $\mathbf{A}^\dagger$, so its restriction to $\mathbf{A}^\#$ is constant on fibres of the projection $\mathbf{A}^\# \to \mathbf{A}^\#/\langle 1_A\rangle$ and defines an affine function $\iota^\#_X$ on the base $\underline{\mathbf{A}^\#} = \mathbf{A}^\#/\langle 1_A\rangle$ of $\mathsf{AV}(\mathbf{A}^\#)$. Hence we have a canonical identification between

1. sections X of $\mathsf{V}(\mathbf{A})$,
2. linear functions $\iota^\dagger_X$ on $\mathbf{A}^\dagger$ which are invariant with respect to the vertical lift of 1_A,
3. affine functions $\iota^\#_X$ on $\underline{\mathbf{A}^\#}$.

In local coordinates and bases, if $X = \sum_1^m f_i(x)e_i$, then $\iota^\dagger_X = \sum_1^{m-1} f_i(x)\xi_i - f_m(x)\xi_m$ and $\iota^\#_X = f_m(x) + \sum_1^{m-1} f_i(x)\xi_i$.

In the theory of vector bundles there is an obvious identification between sections of the bundle E and functions on E^* which are linear along fibres. If φ is a section of E, then the corresponding function ι_φ is defined by the canonical pairing $\iota_\varphi(X) = \langle \varphi, X\rangle$. In the theory of special affine bundles we have an analog of the above identification:

$$\mathsf{Sec}(\mathbf{A}) \simeq \mathsf{AffSec}(\mathsf{AV}(\mathbf{A}^\#)), \qquad \mathbf{F}_\sigma \leftrightarrow \sigma,$$

where $\mathbf{F}_\sigma$ is the unique (affine) function on $\mathbf{A}^\#$ such that $\chi_{\mathsf{AV}(\mathbf{A}^\#)}(\mathbf{F}_\sigma) = 1$ ($\mathbf{F}_\sigma$ can be therefore interpreted as a section of $\mathbf{A}$) and $\mathbf{F}_\sigma \circ \sigma = 0$. In local affine coordinates, we associate with the section

$$\xi_0 = \sigma(x^a, \xi_1, \dots, \xi_{m-1}) = \sum_1^{m-1} \sigma_i(x)\xi_i - \sigma_m(x)$$

of $\mathsf{AV}(\mathbf{A}^\#)$ the function

$$\mathbf{F}_\sigma = \sigma(x^a, \xi_1, \dots, \xi_{m-1}) - \xi_0 = \sum_1^{m-1} \sigma_i(x)\xi_i - \sigma_m(x) - \xi_0$$

on $\mathbf{A}^\#$ which represents the section $M \ni x \mapsto \sum_1^m \sigma_i(x)e_i \in \mathbf{A}$.

Using the above identification we can formulate the following theorem (for the proof see [4]):

Theorem 2.1. *There is a canonical one-to-one correspondence between special Lie affgebroid brackets* $[\cdot,\cdot]_{\mathbf{A}}$ *on* $\mathbf{A}$ *and affine-closed aff-Poisson brackets* $\{\cdot,\cdot\}_{\mathbf{A}^\#}$ *on* $\mathsf{AV}(\mathbf{A}^\#)$*, uniquely defined by:*

$$\{\sigma, \sigma'\}_{\mathbf{A}^\#} = \iota^\#_{[\mathbf{F}_\sigma, \mathbf{F}_{\sigma'}]_{\mathbf{A}}}.$$

Example 2.2. Let $\mathbf{Z} = (Z, v_{\mathbf{Z}})$ be an AV-bundle. The AV-bundle $\mathsf{AV}((\widetilde{\mathsf{T}}\mathbf{Z})^\#)$ is the trivial bundle over the affine phase bundle $\mathsf{P}\mathbf{Z}$ and the aff-Poisson bracket, associated with the canonical special Lie algebroid bracket on $\widetilde{\mathsf{T}}\mathbf{Z}$, is the standard Poisson bracket on $\mathsf{P}\mathbf{Z}$ associated with the canonical symplectic form $\omega_{\mathbf{Z}}$ on $\mathsf{P}\mathbf{Z}$.

3. Double affine bundles

Let M be a smooth manifold and let (x^a), $a = 1, \dots, n$, be a coordinate system in M. We denote by $\tau_M : \mathsf{T}M \to M$ the tangent vector bundle and by $\pi_M : \mathsf{T}^*M \to M$ the cotangent vector bundle. We have the induced (adapted) coordinate systems $(x^a, \dot{x}^b)$ in $\mathsf{T}M$ and (x^a, p_b) in T^*M. Let $\tau : E \to M$ be a vector bundle and let

$\pi\colon E^*\to M$ be the dual bundle. Let $(e_1,\dots,e_m)$ be a basis of local sections of $\tau\colon E\to M$ and let $(e^1_*,\dots,e^m_*)$ be the dual basis of local sections of $\pi\colon E^*\to M$. We have the induced coordinate systems:

$$(x^a,y^i),\quad y^i=\iota(e^i_*),\ \text{in } E,$$
$$(x^a,\xi_i),\quad \xi_i=\iota(e_i),\ \text{in } E^*,$$

where the linear functions $\iota(e)$ are given by the canonical pairing $\iota(e)(v_x)=\langle e(x),v_x\rangle$. Thus we have local coordinates

$$(x^a,y^i,\dot{x}^b,\dot{y}^j)\quad \text{in } \mathsf{T}E,$$
$$(x^a,\xi_i,\dot{x}^b,\dot{\xi}_j)\quad \text{in } \mathsf{T}E^*,$$
$$(x^a,y^i,p_b,\pi_j)\quad \text{in } \mathsf{T}^*E,$$
$$(x^a,\xi_i,p_b,\varphi^j)\quad \text{in } \mathsf{T}^*E^*.$$

It is well known (cf. [11]) that the cotangent bundles T^*E and T^*E^* are examples of double vector bundles:

$$\begin{tikzcd} \mathsf{T}^*E^* \arrow[r, "\mathsf{T}^*\pi"] \arrow[d, "\pi_{E^*}"'] & E \arrow[d, "\tau"] \\ E^* \arrow[r, "\pi"] & M, \end{tikzcd}\qquad \begin{tikzcd} \mathsf{T}^*E \arrow[r, "\mathsf{T}^*\tau"] \arrow[d, "\tau_{E^*}"'] & E^* \arrow[d, "\pi"] \\ E \arrow[r, "\tau"] & M. \end{tikzcd}$$

Note that the concept of a double vector bundle is due to Pradines [23,24], see also [1,11]. In particular, all arrows correspond to vector bundle structures and all pairs of vertical and horizontal arrows are vector bundle morphisms. The above double vector bundles are canonically isomorphic with the isomorphism

$$\mathcal{R}_\tau\colon \mathsf{T}^*E\longrightarrow \mathsf{T}^*E^* \tag{3.1}$$

being simultaneously an anti-symplectomorphism (cf. [2,11,8]). In local coordinates, $\mathcal{R}_\tau$ is given by

$$\mathcal{R}_\tau(x^a,y^i,p_b,\pi_j)=(x^a,\pi_i,-p_b,y^j).$$

Exactly like the cotangent bundle T^*E is a double vector bundle, the affine phase bundle $\mathsf{P}\mathbf{A}$ is canonically a *double affine bundle*:

$$\begin{tikzcd} \mathsf{P}\mathbf{A} \arrow[r, "\mathsf{P}^\#\zeta"] \arrow[d, "\mathsf{P}\zeta"'] & \underline{\mathbf{A}^\#} \arrow[d, "\underline{\eta^\#}"] \\ \underline{\mathbf{A}} \arrow[r, "\underline{\eta}"] & M. \end{tikzcd}$$

Definition. (1) A *trivial double affine bundle* over a base manifold M is a commutative diagram of four affine bundle projections

$$\begin{tikzcd} A \arrow[r, "\Xi_2"] \arrow[d, "\Xi_1"'] & A_2 \arrow[d, "\eta_2"] \\ A_1 \arrow[r, "\eta_1"] & M, \end{tikzcd} \tag{3.2}$$

where $A=M\times K_1\times K_2\times K$ is a trivial affine bundle with the fibers being direct products of (finite-dimensional) affine spaces K_1,K_2,K, with the obvious projections:

$$\Xi_i : M\times K_1\times K_2\times K\to A_i=M\times K_i,\qquad \eta_i : A_i=M\times K_i\to M,\quad i=1,2.$$

In particular, the pairs (Ξ_i,η_i), $i=1,2$, are morphism of affine bundles.

(2) A *morphism of trivial double affine bundles over the identity on the base* is a commutative diagram of morphisms of trivial affine bundles as above

$$\begin{tikzcd}
 & A \arrow[ddl, "\Xi_1"'] \arrow[dr, "\Xi_2"] & & A' \arrow[ll, "\Phi"'] \arrow[ddl, "\Xi'_1"'] \arrow[dr, "\Xi'_2"] & \\
 & & A_2 \arrow[ddl, "\eta_2"] & & A'_2 \arrow[ll, "\Phi_2"'] \arrow[ddl, "\eta'_2"] \\
A_1 \arrow[dr, "\eta_1"'] & & A'_1 \arrow[ll, "\Phi_1"'] \arrow[dr, "\eta'_1"] & & \\
 & M & & M \arrow[ll, "id"'] &
\end{tikzcd} \tag{3.3}$$

where clearly $A = M \times K_1 \times K_2 \times K$, $A' = M \times K'_1 \times K'_2 \times K'$, etc., and (Ξ_i, η_i), (Ξ'_i, η'_i), (Φ, Φ_i), and (Φ_i, id), $i = 1, 2$, are morphisms of affine bundles. Note that we have not considered A as an affine bundle over M, since this structure is accidental in the trivial case and it is not respected by isomorphisms in the above sense.

(3) A *double affine bundle* with the model fibre being the product $K_1 \times K_2 \times K$ of affine spaces is a commutative diagram (3.2) of affine bundles, this time not necessarily trivial, which is locally diffeomorphic with trivial double affine bundles associated with $K_1 \times K_2 \times K$, i.e. there is an open covering $\{U_\alpha\}$ of M such that

$$\begin{tikzcd}
A^\alpha \arrow[r, "\Xi_{2|A^\alpha}"] \arrow[d, "\Xi_{1|A^\alpha}"'] & A_2^\alpha \arrow[d, "\eta_{2|A_2^\alpha}"] \\
A_1^\alpha \arrow[r, "\eta_{1|A_1^\alpha}"] & U_\alpha,
\end{tikzcd}$$

where $A_i^\alpha = \eta_i^{-1}(U_\alpha)$, $i = 1, 2$, and $A^\alpha = (\Xi_1 \circ \eta_1)^{-1}(U_\alpha) = (\Xi_2 \circ \eta_2)^{-1}(U_\alpha)$, is (diffeomorphically) equivalent to the trivial double affine bundle

$$\begin{tikzcd}
U_\alpha \times K_1 \times K_2 \times K \arrow[r] \arrow[d] & U_\alpha \times K_2 \arrow[d] \\
U_\alpha \times K_1 \arrow[r] & U_\alpha
\end{tikzcd}$$

and such that this equivalence induces an automorphism of the corresponding trivial double affine bundle over the identity on each intersection $U_\alpha \bigcap U_\beta$.

(4) A *morphism of double affine bundles* is a commutative diagram (3.3) of morphisms of corresponding affine bundles which in local trivializations induces morphisms of trivial double affine bundles.

To see how the trivial double affine bundles are glued up inside a (non-trivial) double affine bundle, let us take affine coordinates (x, y, z, c) in $A = \mathbb{R}^k \times K_1 \times K_2 \times K$. Then every morphism $\Phi : A \to A$ of the double affine bundle into itself (over the identity on the base) is of the form

$$\begin{aligned}\Phi(x, y^j, z^a, c^u) = \Bigg(& x, \alpha_0^j(x) + \sum_i \alpha_i^j(x) y^i, \beta_0^a(x) + \sum_b \beta_b^a(x) z^b, \gamma_{00}^u(x) \\ & + \sum_i \gamma_{i0}^u(x) y^i + \sum_b \gamma_{0b}^u(x) z^b + \sum_{i,b} \gamma_{ib}^u(x) y^i z^b + \sum_w \sigma_w^u(x) c^w \Bigg).\end{aligned}$$

Let $\mathbf{A}$ be a special affine bundle. The adapted affine coordinates on $\mathsf{P}\mathbf{A}$ are (x^a, y^i, p_b, ξ_j), $i, j = 1, \ldots, m-1$, and the projections read

$$\mathsf{P}^{\#}\zeta(x, y, p, \xi) = (x, \xi), \qquad \underline{\eta}^{\#}(x, y, p, \xi) = (x, y).$$

The canonical symplectic form reads $\omega_{\mathsf{P}\mathbf{A}} = \sum_a \mathrm{d}p_a \wedge \mathrm{d}x^a + \sum_i \mathrm{d}\xi_i \wedge \mathrm{d}y^i$ and the affine de Rham differential $\mathbf{d} : \mathsf{Sec}(\mathsf{AV}(\mathbf{A})) \to \mathsf{Sec}(\mathsf{P}\mathbf{A})$ associates with a section $y^m = \sigma(x, y^i)$, $i = 1, \ldots, m-1$ the section $\mathbf{d}\sigma$:

$$p_a \circ \mathbf{d}\sigma = \frac{\partial \sigma}{\partial x^a}, \qquad \xi_i \circ \mathbf{d}\sigma = \frac{\partial \sigma}{\partial y^i}.$$

The special vector bundle $\widetilde{\mathsf{T}}(\mathsf{AV}(\mathbf{A})) = (\mathsf{P}\mathbf{A})^{\dagger}$, which will be denoted shortly $\widetilde{\mathsf{T}}\mathbf{A}$, carries the coordinates $(x^a, y^i, \dot{x}^b, \dot{y}^j, s)$, $i, j = 1, \ldots, m-1$, with the affine-linear pairing $\langle \cdot, \cdot \rangle_{\dagger} : \mathsf{P}\mathbf{A} \times_{\underline{\mathbf{A}}} \widetilde{\mathsf{T}}\mathbf{A} \to \mathbb{R}$ given by

$$\langle (x, y, p, \xi), (x, y, \dot{x}, \dot{y}, s) \rangle_{\dagger} = \sum_i \xi_i \dot{y}^i + \sum_a \dot{x}^a p_a - s.$$

We have the obvious projection $\widetilde{\mathsf{T}}\mathbf{A} \to \mathsf{T}\underline{\mathbf{A}}$ that reads $(x, y, \dot{x}, \dot{y}, s) \mapsto (x, y, \dot{x}, \dot{y})$. Similarly to the linear case, the double affine bundles $\mathsf{P}\mathbf{A}$ and $\mathsf{P}\mathbf{A}^{\#}$ are canonically isomorphic (see [31])

$$\begin{tikzcd}
 & \mathsf{P}\mathbf{A}^{\#} \arrow[ddl, "\mathsf{P}\eta^{\#}"'] \arrow[dr, "\mathsf{P}^{\#}\eta^{\#}"] & & \mathsf{P}\mathbf{A} \arrow[ll, "\mathcal{R}_\eta"'] \arrow[ddl, "\mathsf{P}^{\#}\eta"'] \arrow[dr, "\mathsf{P}\eta"] & \\
 & & \underline{\mathbf{A}} \arrow[ddl, "\underline{\eta}"'] & & \underline{\mathbf{A}} \arrow[ll, "id"'] \arrow[ddl, "\underline{\eta}"] \\
\underline{\mathbf{A}}^{\#} \arrow[dr, "\underline{\eta}^{\#}"] & & \underline{\mathbf{A}}^{\#} \arrow[ll, "id"'] \arrow[dr, "\underline{\eta}^{\#}"] & & \\
 & M & & M \arrow[ll, "id"'] &
\end{tikzcd} \qquad (3.4)$$

In local coordinates, $\mathcal{R}_\eta$ is given by

$$\mathcal{R}_\eta(x^a, y^i, p_b, \xi_j) = (x^a, \xi_i, -p_b, y^j).$$

4. Lie algebroids as double vector bundle morphisms

For Lie algebroids we refer to the survey article [13]. It is well known that Lie algebroid structures on a vector bundle E correspond to linear Poisson tensors on E^*. A 2-contravariant tensor Π on E^* is called *linear* if the corresponding mapping $\widetilde{\Pi} : \mathsf{T}^*E^* \to \mathsf{T}E^*$ induced by contraction is a morphism of double vector bundles. This is the same as to say that the corresponding bracket of functions is closed on (fiber-wise) linear functions. The commutative diagram

$$\begin{tikzcd}
\mathsf{T}^*E^* \arrow[r, "\widetilde{\Pi}"] & \mathsf{T}E^* \\
\mathsf{T}^*E \arrow[u, "\mathcal{R}_\tau"] \arrow[ur, "\varepsilon"'] &
\end{tikzcd},$$

composed with (3.1), describes a one-to-one correspondence between linear 2-contravariant tensors Π on E^* and homomorphisms of double vector bundles (cf. [11,8]) covering the identity on E^*:

$$
\begin{tikzcd}
 & \mathsf{T}^*E \arrow[rrr, "\varepsilon"] \arrow[dr, "\pi_E"] \arrow[ddl, "\mathsf{T}^*\tau"'] & & & \mathsf{T}E^* \arrow[dr, "\mathsf{T}\pi"] \arrow[ddl, "\tau_{E^*}"'] & \\
 & & E \arrow[rrr, "\varepsilon_l"] \arrow[ddl, "\tau"'] & & & \mathsf{T}M \arrow[ddl, "\tau_M"'] \\
E^* \arrow[rrr, "id"] \arrow[dr, "\pi"] & & & E^* \arrow[dr, "\pi"] & & \\
 & M \arrow[rrr, "id"] & & & M &
\end{tikzcd}
\tag{4.1}
$$

In local coordinates, every ε as above is of the form

$$
\varepsilon(x^a, y^i, p_b, \pi_j) = \left(x^a, \pi_i, \sum_k \rho_k^b(x) y^k, \sum_{i,k} c_{ij}^k(x) y^i \pi_k + \sum_a \sigma_j^a(x) p_a\right) \tag{4.2}
$$

and it corresponds to the linear tensor

$$
\Pi_\varepsilon = \sum_{i,j,k} c_{ij}^k(x) \xi_k \partial_{\xi_i} \otimes \partial_{\xi_j} + \sum_{i,b} \rho_i^b(x) \partial_{\xi_i} \otimes \partial_{x^b} - \sum_{a,j} \sigma_j^a(x) \partial_{x^a} \otimes \partial_{\xi_j}.
$$

In [8] by *algebroids* we meant the morphisms (4.1) of double vector bundles covering the identity on E^*, while *Lie algebroids* were those algebroids for which the tensor Π_ε is a Poisson tensor. The relation to the canonical definition of Lie algebroid is given by the following theorem (cf. [7,8]).

Theorem 4.1. *An algebroid structure (E, ε) can be equivalently defined as a bilinear bracket $[\cdot,\cdot]_\varepsilon$ on sections of $\tau\colon E \to M$, together with vector bundle morphisms $\varepsilon_l,\ \varepsilon_r\colon E \to \mathsf{T}M$ (left and right anchors), such that*

$$
[fX, gY]_\varepsilon = f(\varepsilon_l \circ X)(g)Y - g(\varepsilon_r \circ Y)(f)X + fg[X,Y]_\varepsilon
$$

for $f, g \in \mathcal{C}^\infty(M)$, $X, Y \in \otimes^1(\tau)$. The bracket and anchors are related to the 2-contravariant tensor Π_ε by the formulae

$$
\begin{aligned}
&\iota([X,Y]_\varepsilon) = \{\iota(X), \iota(Y)\}_{\Pi_\varepsilon},\\
&\pi^*(\varepsilon_l \circ X(f)) = \{\iota(X), \pi^* f\}_{\Pi_\varepsilon},\\
&\pi^*(\varepsilon_r \circ X(f)) = \{\pi^* f, \iota(X)\}_{\Pi_\varepsilon}.
\end{aligned}
$$

The algebroid (E, ε) is a Lie algebroid if and only if the tensor Π_ε is a Poisson tensor.

The canonical example of a mapping ε in the case of $E = \mathsf{T}M$ is given by $\varepsilon = \varepsilon_M = \alpha_M^{-1}$ — the inverse to the Tulczyjew isomorphism $\alpha_M : \mathsf{T}\mathsf{T}^*M \to \mathsf{T}^*\mathsf{T}M$ [29]. In general, the algebroid structure map ε is not an isomorphism and, consequently, its dual $\kappa^{-1} = \varepsilon^{*r}$ with respect to the right projection is a relation and not a mapping.

5. Lagrangian and Hamiltonian formalisms for general algebroids

The double vector bundle morphism (4.1) can be extended to the following algebroid analog of the so called Tulczyjew triple

$$\begin{tikzcd}
 & \mathsf{T}^*E^* \arrow[rrr, "\widetilde{\Pi}"] \arrow[dr, "\mathsf{T}^*\pi"] \arrow[ddl, "\pi_{E^*}"'] & & & \mathsf{T}E^* \arrow[ddl, "\tau_{E^*}"'] \arrow[dr, "\mathsf{T}\pi"] & & & \mathsf{T}^*E \arrow[lll, "\varepsilon"'] \arrow[ddl, "\mathsf{T}^*\tau"'] \arrow[dr, "\pi_E"] & \\
 & & E \arrow[rrr, "\widetilde{\Pi}_r"] \arrow[ddl, "\tau"'] & & & \mathsf{T}M \arrow[ddl, "\tau_M"'] & & & E \arrow[lll, "\varepsilon_l"'] \arrow[ddl, "\tau"'] \\
E^* \arrow[rrr, "id"] \arrow[dr, "\pi"] & & & E^* \arrow[dr, "\pi"] & & & E^* \arrow[lll, "id"'] \arrow[dr, "\pi"] & & \\
 & M \arrow[rrr, "id"] & & & M & & & M \arrow[lll, "id"'] & .
\end{tikzcd} \tag{5.1}$$

The left-hand side is Hamiltonian, the right-hand side is Lagrangian, and the dynamics lives in the middle.

The Lagrangian $L : E \to \mathbb{R}$ defines two smooth maps: the *Legendre mapping*: $\lambda_L : E \longrightarrow E^*$, $\lambda_L = \tau_{E^*} \circ \varepsilon \circ \mathrm{d}L$, which is covered by the *Tulczyjew differential* $\Lambda_L : E \longrightarrow \mathsf{T}E^*$, $\Lambda_L = \varepsilon \circ \mathrm{d}L$:

$$\begin{tikzcd}
\mathsf{T}^*E \arrow[r, "\varepsilon"] & \mathsf{T}E^* \arrow[d, "\tau_{E^*}"] \\
E \arrow[u, "\mathrm{d}L"] \arrow[ur, "\Lambda_L"] \arrow[r, "\lambda_L"] & E^*.
\end{tikzcd}$$

The lagrangian function defines *the phase dynamics* $D = \Lambda_L(E) \subset \mathsf{T}E^*$ which can be understood as an implicit differential equation on E^*, solutions of which are 'phase trajectories' of the system, i.e. curves $\beta : \mathbb{R} \to E^*$ such that $\mathsf{T}\beta(t) \in D$. An equation for curves $\gamma : \mathbb{R} \to E$ (analog of the Euler–Lagrange equation) is:

$$(E_L): \quad \mathsf{T}(\lambda_L \circ \gamma) = \Lambda_L \circ \gamma.$$

In local coordinates, D has the parametrization by (x^a, y^k) via Λ_L in the form (cf. (4.2))

$$\Lambda_L(x^a, y^i) = \left(x^a, \frac{\partial L}{\partial y^i}(x, y), \sum_k \rho^b_k(x) y^k, \sum_{i,k} c^k_{ij}(x) y^i \frac{\partial L}{\partial y^k}(x, y) + \sum_a \sigma^a_j(x) \frac{\partial L}{\partial x^a}(x, y)\right) \tag{5.2}$$

and the equation (E_L), for $\gamma(t) = (x^a(t), y^i(t))$, reads

$$(E_L): \quad \frac{\mathrm{d}x^a}{\mathrm{d}t} = \sum_k \rho^a_k(x) y^k, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^j}\right) = \sum_{i,k} c^k_{ij}(x) y^i \frac{\partial L}{\partial y^k}(x, y) + \sum_a \sigma^a_j(x) \frac{\partial L}{\partial x^a}(x, y), \tag{5.3}$$

in the full agreement with [12,16,17,32], if only one takes into account that, for Lie algebroids, $\sigma^a_j = \rho^a_j$. As one can see from (5.3), the solutions are automatically admissible curves in E, i.e. the velocity $\frac{\mathrm{d}}{\mathrm{d}t}(\tau \circ \gamma)(t)$ equals $\varepsilon_l(\gamma(t))$.

Note that the tensor Π_ε gives rise also to kind of a Hamiltonian formalism (cf. [22]). In [8] and [22] one refers to a 2-contravariant tensor as to a *Leibniz structure*. In the presence of Π_ε, by the *Hamiltonian vector field* associated with a function H on E^* we understand the contraction $\mathrm{i}_{\mathrm{d}H}\Pi_\varepsilon$. Thus the question of the Hamiltonian description of the dynamics D in the simplest form is the question if D is the image of a Hamiltonian vector field. Every such a function H we call a *Hamiltonian associated with the Lagrangian* L. In the case of a hyperregular Lagrangian, i.e. when Λ_L is a diffeomorphism, we recover the standard correspondence between Lagrangians and Hamiltonians (see [5, Corollary 1]).

Theorem 5.1. *If the Lagrangian* L *is hyperregular, then* $H(e^*_x) = \langle \lambda_L^{-1}(e^*_x), e^*_x\rangle - L \circ \lambda_L^{-1}(e^*_x)$ *is a Hamiltonian associated with* L. *This Hamiltonian has the property that the Lagrange submanifold* $N = \mathrm{d}L(E)$ *in* T^*E *corresponds under the canonical isomorphism* $\mathcal{R}_\tau$ *to the Lagrange submanifold* $\mathrm{d}H(E^*)$ *in* T^*E^*.

6. Special affgebroids as morphisms of double affine bundles

Let $\eta : \mathbf{A} = (A, v_A) \to M$ be a special affine bundle. With an analogy to the linear case, by a general *special affgebroid on* $\mathbf{A}$ we mean a morphism of double affine bundles covering the identity on $\underline{\mathbf{A}^\#}$:

$$\begin{tikzcd}
\mathsf{P}\mathbf{A} \arrow[rr, "\mathcal{E}"] \arrow[dr, "\mathsf{P}\eta"] \arrow[dd, "\mathsf{P}^\#\eta"'] & & \mathsf{T}\underline{\mathbf{A}^\#} \arrow[dr, "\mathsf{T}\eta^\#"] \arrow[dd, "\tau_{\underline{\mathbf{A}^\#}}"'] & \\
& \underline{\mathbf{A}} \arrow[rr, "\mathcal{E}_l"] \arrow[dd, "\underline{\eta}"'] & & \mathsf{T}M \arrow[dd, "\tau_M"] \\
\underline{\mathbf{A}^\#} \arrow[rr, "id"] \arrow[dr, "\underline{\eta^\#}"'] & & \underline{\mathbf{A}^\#} \arrow[dr, "\underline{\eta^\#}"'] & \\
& M \arrow[rr, "id"] & & M
\end{tikzcd} \quad . \tag{6.1}$$

Every such morphism is the composition of (3.4) with a morphisms of double affine bundles

$$\begin{tikzcd}
\mathsf{P}\mathbf{A}^\# \arrow[rr, "\widetilde{\Pi}"] \arrow[dr, "\mathsf{P}^\#\eta^\#"] \arrow[dd, "\mathsf{P}\eta^\#"'] & & \mathsf{T}\underline{\mathbf{A}^\#} \arrow[dr, "\mathsf{T}\eta^\#"] \arrow[dd, "\tau_{\underline{\mathbf{A}^\#}}"'] & \\
& \underline{\mathbf{A}} \arrow[rr, "\widetilde{\Pi}_l"] \arrow[dd, "\underline{\eta}"'] & & \mathsf{T}M \arrow[dd, "\tau_M"] \\
\underline{\mathbf{A}^\#} \arrow[rr, "id"] \arrow[dr, "\underline{\eta^\#}"'] & & \underline{\mathbf{A}^\#} \arrow[dr, "\underline{\eta^\#}"'] & \\
& M \arrow[rr, "id"] & & M
\end{tikzcd} \quad . \tag{6.2}$$

Such morphisms correspond to affine tensors $\Pi = \Pi_{\mathcal{E}} \in \mathsf{Sec}(\widetilde{\mathsf{T}}\mathbf{A}^\# \otimes_{\underline{\mathbf{A}^\#}} \mathsf{T}\underline{\mathbf{A}^\#})$ or, equivalently, to affine-closed biderivation brackets

$$\{\cdot,\cdot\}_\Pi : \mathsf{Sec}(\mathsf{AV}(\mathbf{A}^\#)) \times C^\infty(\underline{\mathbf{A}^\#}) \to C^\infty(\underline{\mathbf{A}^\#}). \tag{6.3}$$

This obvious equivalence is induced by the standard identification

$$\widetilde{\mathsf{T}}\mathbf{A}^\# \otimes_{\underline{\mathbf{A}^\#}} \mathsf{T}\underline{\mathbf{A}^\#} \simeq \mathsf{Hom}_{\underline{\mathbf{A}^\#}}((\widetilde{\mathsf{T}}\mathbf{A}^\#)^*, \mathsf{T}\underline{\mathbf{A}^\#}) \simeq \mathsf{Hom}_{\underline{\mathbf{A}^\#}}(\widehat{\mathsf{P}\mathbf{A}^\#}, \mathsf{T}\underline{\mathbf{A}^\#}) \simeq \mathsf{Aff}_{\underline{\mathbf{A}^\#}}(\mathsf{P}\mathbf{A}^\#, \mathsf{T}\underline{\mathbf{A}^\#}),$$

so

$$\{\sigma, f\}_\Pi = \langle \Pi, \mathbf{d}\sigma \otimes \mathrm{d}f \rangle,$$

where the affine de Rham differential $\mathbf{d}\sigma \in \mathsf{Sec}(\mathsf{P}\mathbf{A}^\#)$ is regarded also as a section of the vector hull $\widehat{\mathsf{P}\mathbf{A}^\#}$. That (6.3) is an affine-closed biderivation means that the bracket is an (affine) derivation with respect to the first argument, a derivation with respect to the second argument, and it is affine-closed, i.e. the bracket $\{\sigma, f\}_\Pi$ of an affine section $\sigma : \underline{\mathbf{A}^\#} \to \mathbf{A}^\#$ and an affine function f on $\underline{\mathbf{A}^\#}$ is an affine function on $\underline{\mathbf{A}^\#}$.

Note that such brackets are just affine-linear parts of certain affine-closed biderivations

$$\{\cdot,\cdot\} : \mathsf{Sec}(\mathsf{AV}(\mathbf{A}^\#)) \times \mathsf{Sec}(\mathsf{AV}(\mathbf{A}^\#)) \to C^\infty(\underline{\mathbf{A}^\#}).$$

On the level of tensors it means that Π can be understood as the projection of a tensor from $\mathsf{Sec}(\widetilde{\mathsf{T}}\mathbf{A}^\# \otimes_{\underline{\mathbf{A}^\#}} \widetilde{\mathsf{T}}\mathbf{A}^\#)$, i.e. as the projection of a $\chi_{\mathbf{A}}$-invariant and affine 2-contravariant tensor on $\mathbf{A}^\#$. Note that in the skew-symmetric case, e.g. in the case of an aff-Poisson bracket, there is a one-to-one correspondence between skew biderivations and their affine-linear parts, since $\{a, b\} = \{a, b - a\}^2_{\mathsf{v}}$, where the latter is the linear part with respect to the second argument.

According to the identification of sections of $\mathbf{A}$ with affine sections of $\mathsf{AV}(\mathbf{A}^{\#})$ [4, Theorem 13] we can derive out of the bracket $\{\cdot,\cdot\}_{\Pi}$, similarly as it has been done for aff-Poisson brackets, a general *special affgebroid bracket*

$$[\cdot,\cdot]_{\Pi}:\mathsf{Sec}(\mathbf{A})\times\mathsf{Sec}(\mathsf{V}(\mathbf{A}))\to\mathsf{Sec}(\mathsf{V}(\mathbf{A})).$$

The following theorem (which is completely analogous to [4, Theorem 23], so we skip the proof) explains in details what we understand as a special affgebroid bracket.

Theorem 6.1. *A special affgebroid structure (6.1) can be equivalently defined as an affine-linear bracket*

$$[\cdot,\cdot]_{\mathcal{E}}:\mathsf{Sec}(\mathbf{A})\times\mathsf{Sec}(\mathsf{V}(\mathbf{A}))\to\mathsf{Sec}(\mathsf{V}(\mathbf{A}))$$

which is special (i.e. $[a,u+v_{\mathbf{A}}]_{\mathcal{E}}=[a+v_{\mathbf{A}},u]_{\mathcal{E}}=[a,u]_{\mathcal{E}}$), together with an affine bundle morphisms $\mathcal{E}_l: A\to\mathsf{T}M$ and a vector bundle morphism $\mathcal{E}_r:\mathsf{V}(A)\to\mathsf{T}M$ (left and right anchors), such that

$$[a,gY]_{\mathcal{E}}=g[a,Y]_{\mathcal{E}}+(\mathcal{E}_l\circ a)(g)Y$$

and

$$[a+fX,Y]_{\mathcal{E}}=(1-f)[a,Y]_{\mathcal{E}}+f[a+X,Y]_{\mathcal{E}}-(\mathcal{E}_r\circ Y)(f)X$$

where a is a section of $\mathbf{A}$ and X is a section of $\mathsf{V}(\mathbf{A})$. The brackets $[\cdot,\cdot]_{\mathcal{E}}$, $\{\cdot,\cdot\}_{\mathcal{E}}$ and the tensor $\Pi=\Pi_{\mathcal{E}}\in\mathsf{Sec}(\widetilde{\mathsf{T}}\mathbf{A}^{\#}\otimes\mathsf{T}\underline{\mathbf{A}^{\#}})$ are related by the formula

$$\langle\Pi_{\mathcal{E}},\mathbf{dF}_a\otimes\mathrm{d}\iota_X\rangle=\iota^{\#}_{[a,X]_{\mathcal{E}}}=\{\mathbf{F}_a,\iota^{\#}_X\}_{\mathcal{E}},$$

where $\mathbf{F}_a$ (resp., $\iota^{\#}_X$) is the corresponding section of $\mathsf{AV}(\mathbf{A}^{\#})$ (resp., the corresponding function on $\underline{\mathbf{A}^{\#}}$). The special affgebroid is a special Lie affgebroid if and only if the tensor $\Pi_{\mathcal{E}}$ is an aff-Poisson tensor.

In local affine coordinates, every $\mathcal{E}$ as above is of the form

$$\begin{aligned}\mathcal{E}(x^a,y^i,p_b,\pi_j)=\Bigg(&x^a,\pi_i,\rho^b_0(x)+\sum_k\rho^b_k(x)y^k,c^m_{0j}(x)+\sum_k c^k_{0j}(x)\pi_k\\&+\sum_i c^m_{ij}(x)y^i+\sum_{i,k}c^k_{ij}(x)y^i\pi_k+\sum_a\sigma^a_j(x)p_a\Bigg),\end{aligned}\tag{6.4}$$

where $i,j,k=1,\ldots,m-1$, and $\mathcal{E}$ corresponds to the affine 2-contravariant tensor $\Pi_{\mathcal{E}}$ on $\mathbf{A}^{\#}$

$$\Pi_{\mathcal{E}}=\sum_{i=0,j=1}^{m-1}\left(c^m_{ij}(x)+\sum_{k=1}^{m-1}c^k_{ij}(x)\xi_k\right)\partial_{\xi_i}\otimes\partial_{\xi_j}+\sum_b\left(\sum_{i=0}^{m-1}\rho^b_i(x)\partial_{\xi_i}\otimes\partial_{x^b}-\sum_{j=1}^{m-1}\sigma^b_j(x)\partial_{x^b}\otimes\partial_{\xi_j}\right).\tag{6.5}$$

The corresponding affgebroid bracket on

$$\mathsf{Sec}(\mathbf{A})\times\mathsf{Sec}(\mathsf{V}(\mathbf{A}))\subset\mathsf{Sec}(\widehat{\mathbf{A}})\times\mathsf{Sec}(\widehat{\mathbf{A}})$$

reads

$$\left[e_0+\sum_{i=1}^m f_ie_i,\sum_{j=1}^m g_je_j\right]=\sum_{k=1}^m\left(\sum_{i=0,j=1}^{m-1}f_ig_jc^k_{ij}+\sum_a\left(\sum_{i=0}^{m-1}\rho^a_if_i\frac{\partial g_k}{\partial x^a}-\sum_{j=1}^{m-1}\sigma^a_jg_j\frac{\partial f_k}{\partial x^a}\right)\right)e_k,$$

with the convention that $f_0=1$.

7. Lagrangian and Hamiltonian formalisms for special affine bundles

Combining (6.1) and (6.2) we get the *affine Tulczyjew triple*:

$$\begin{tikzcd}
 & \mathsf{P}\mathbf{A}^\# \arrow[rrr, "\widetilde{\Pi}"] \arrow[dr, "\mathsf{P}^\#\eta^\#"] \arrow[ddl, "\mathsf{P}\eta^\#"'] & & & \mathsf{T}\underline{\mathbf{A}}^\# \arrow[ddl, "\tau_{\underline{\mathbf{A}}^\#}"'] \arrow[dr, "\mathsf{T}\eta^\#"] & & & \mathsf{P}\mathbf{A} \arrow[lll, "\mathcal{E}"'] \arrow[ddl, "\mathsf{P}^\#\eta"'] \arrow[dr, "\mathsf{P}\eta"] & \\
 & & \underline{\mathbf{A}} \arrow[rrr, "\widetilde{\Pi}_r"] \arrow[ddl, "\underline{\eta}"] & & & \mathsf{T}M \arrow[ddl, "\tau_M"] & & & \underline{\mathbf{A}} \arrow[lll, "\mathcal{E}_l"'] \arrow[ddl, "\underline{\eta}"] \\
\underline{\mathbf{A}}^\# \arrow[rrr, "id"] \arrow[dr, "\underline{\eta}^\#"'] & & & \underline{\mathbf{A}}^\# \arrow[dr, "\underline{\eta}^\#"'] & & & \underline{\mathbf{A}}^\# \arrow[lll, "id"'] \arrow[dr, "\underline{\eta}^\#"'] & & \\
 & M \arrow[rrr, "id"] & & & M & & & M \arrow[lll, "id"'] &
\end{tikzcd} \tag{7.1}$$

The left-hand side is Hamiltonian, the right-hand side is Lagrangian, and the 'dynamics' lives in the middle.

The Lagrangian section $\mathcal{L} : \underline{\mathbf{A}} \to \mathbf{A}$ defines also smooth maps:

$$\begin{tikzcd}
\mathsf{P}\mathbf{A} \arrow[r, "\mathcal{E}"] & \mathsf{T}\underline{\mathbf{A}}^\# \arrow[d, "\tau_{\underline{\mathbf{A}}^\#}"] \\
\underline{\mathbf{A}} \arrow[u, "\mathrm{d}\mathcal{L}"] \arrow[ur, "\Lambda_{\mathcal{L}}"] \arrow[r, "\lambda_{\mathcal{L}}"] & \underline{\mathbf{A}}^\#.
\end{tikzcd}$$

The map $\lambda_{\mathcal{L}} : \underline{\mathbf{A}} \to \underline{\mathbf{A}}^\#$, $\lambda_{\mathcal{L}} = \tau_{\underline{\mathbf{A}}^\#} \circ \mathcal{E} \circ \mathbf{d}\mathcal{L}$ — the *affine Legendre mapping* associated with $\mathcal{L}$, and the *affine Tulczyjew differential* $\Lambda_{\mathcal{L}} : \underline{\mathbf{A}} \to \mathsf{T}\underline{\mathbf{A}}^\#$, $\Lambda_{\mathcal{L}} = \mathcal{E} \circ \mathbf{d}\mathcal{L}$.

The Lagrangian section defines *the phase dynamics*

$$\mathcal{D} = \Lambda_{\mathcal{L}}(\underline{\mathbf{A}}) \subset \mathsf{T}\underline{\mathbf{A}}^\#,$$

whose integral curves $\beta : \mathbb{R} \to \underline{\mathbf{A}}^\#$ satisfy $\mathsf{T}\beta(t) \in \mathcal{D}$. An equation for curves $\gamma : \mathbb{R} \to \underline{\mathbf{A}}$ (analog of the Euler–Lagrange equation) is:

$$(E_{\mathcal{L}}): \quad \mathsf{T}(\lambda_{\mathcal{L}} \circ \gamma) = \Lambda_{\mathcal{L}} \circ \gamma.$$

This equation is represented by a subset $E_{\mathcal{L}}$ of $\mathsf{T}\underline{\mathbf{A}}$ being the inverse image

$$E_{\mathcal{L}} = \mathsf{T}(\widetilde{\mathcal{L}}_{eg})^{-1}(\mathsf{T}^2\underline{\mathbf{A}}^\#)$$

of the subbundle $\mathsf{T}^2\underline{\mathbf{A}}^\#$ of holonomic vectors in $\mathsf{T}\mathsf{T}\underline{\mathbf{A}}^\#$.

In local coordinates $\mathcal{L}$ is just a function $\mathcal{L} = \mathcal{L}(x, y)$ and $\mathcal{D}$ has the parametrization by (x^a, y^i) via $\Lambda_{\mathcal{L}}$ in the form (cf. (6.4))

$$\begin{aligned}\Lambda_{\mathcal{L}}(x^a, y^i) = \Bigg(& x^a, \frac{\partial \mathcal{L}}{\partial y^i}(x, y), \rho_0^b(x) + \sum_k \rho_k^b(x) y^k, c_{0j}^m(x) + \sum_i c_{ij}^m(x) y^i \\ & + \sum_k \frac{\partial \mathcal{L}}{\partial y^k}(x, y) \left(c_{0j}^k(x) + \sum_i c_{ij}^k(x) y^i \right) + \sum_a \sigma_j^a(x) \frac{\partial \mathcal{L}}{\partial x^a}(x, y) \Bigg),\end{aligned} \tag{7.2}$$

and the equation $(E_{\mathcal{L}})$, for $\gamma(t) = (x^a(t), y^i(t))$, is the system of equations

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = \rho_0^a(x) + \sum_k \rho_k^a(x) y^k, \tag{7.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial y^j}\right) = c_{0j}^m(x) + \sum_i c_{ij}^m(x) y^i + \sum_k \frac{\partial \mathcal{L}}{\partial y^k}\left(c_{0j}^k(x) + \sum_i c_{ij}^k(x) y^i \right) + \sum_a \sigma_j^a(x) \frac{\partial \mathcal{L}}{\partial x^a}. \tag{7.4}$$

Note that in the particular case when the special affine bundle is trivial, $\mathbf{A} = A_0 \times \mathbb{R}$, and the special affgebroid structure on $\mathbf{A}$ comes from the product of a Lie affgebroid structure on A_0 and the trivial Lie algebroid structure in $\mathbb{R}$, this is in the full agreement with [9, (3.14)], if only one takes into account that in this case $\sigma^a_j = \rho^a_j$ and $c^m_{ij} = 0$. As one can see from (7.3), the solutions are automatically admissible curves in $\underline{\mathbf{A}}$, i.e. the velocity $\frac{\mathrm{d}}{\mathrm{d}t}(\underline{\eta} \circ \gamma)(t)$ equals $\mathcal{E}_l(\gamma(t))$.

Like in the algebroid case, the Hamiltonian formalism is related to the tensor $\Pi_{\mathcal{E}}$ (or the aff-Poisson bracket $\{\cdot,\cdot\}_{\mathcal{E}}$ on $\mathsf{AV}(\mathbf{A}^\#)$). By the *Hamiltonian vector field* associated with a section $\mathcal{H}$ of $\mathsf{AV}(\mathbf{A}^\#)$ we understand the vector field on $\underline{\mathbf{A}}^\#$ associated with the derivation $\{\mathcal{H},\cdot\}_{\mathcal{E}}$ of $C^\infty(\underline{\mathbf{A}}^\#)$. Thus the question of the Hamiltonian description of the dynamics $\mathcal{D}$ in the simplest form is the question if $\mathcal{D}$ is the image of a Hamiltonian vector field. Every such a section $\mathcal{H}$ we call a *Hamiltonian associated with the Lagrangian* $\mathcal{L}$.

Like in the algebroid case (cf. Theorem 5.1), when dealing with a hyperregular Lagrangian section, i.e. when $\lambda_{\mathcal{L}}$ is a diffeomorphism, we can find a Hamiltonian associated with the Lagrangian $\mathcal{L}$ explicitly. To describe this "affine Legendre transformation" let us notice that with every section $\mathcal{L}$ of $\mathsf{AV}(\mathbf{A})$ we can associate a map $\widehat{\mathcal{L}} : \underline{\mathbf{A}} \to \mathbf{A}^\#$ as follows. Let us fix $x \in M$ and $a_x \in (\underline{\mathbf{A}})_x$ and let W_{a_x} be the maximal affine subspace in A_x that is tangent to the submanifold $\mathcal{L}((\underline{\mathbf{A}})_x)$ at $\mathcal{L}(a_x)$. There is a unique affine function $\widehat{\mathcal{L}}_{a_x}$ on A_x which is from $\mathbf{A}^\#_x$ (i.e. $\chi_{\mathbf{A}}(\widehat{\mathcal{L}}(a_x)) = -1$) and which vanishes on W_{a_x}.

Theorem 7.1. *If the Lagrangian section $\mathcal{L}$ is hyperregular, then $\mathcal{H} = \widehat{\mathcal{L}} \circ \Lambda^{-1}_{\mathcal{L}}$ is a section of $\mathsf{AV}(\mathbf{A}^\#)$ which is a Hamiltonian associated with $\mathcal{L}$.*

Proof. Let us use local coordinates and the pairing $\langle\cdot,\cdot\rangle_{sa} : \mathbf{A} \times_M \mathbf{A}^\# \to \mathbb{R}$ as in (2.5). Since the distinguished direction in $\mathbf{A}$ is $-\partial_{y^m}$, it is easy to see that the affine function $\widehat{\mathcal{L}}_{a_x}$, $a_x = (x, y^1_0, \ldots, y^{m-1}_0)$, is

$$\widehat{\mathcal{L}}_{a_x}(x, y^1, \ldots, y^{m-1}) = \left(\sum_{i=1}^{m-1}(y^i - y^i_0)\frac{\partial \mathcal{L}}{\partial y^i}(a_x) + \mathcal{L}(a_x)\right) - y^m,$$

which corresponds to the element

$$\left(x, \sum_{i=1}^{m-1} y^i_0 \frac{\partial \mathcal{L}}{\partial y^i}(a_x) - \mathcal{L}(a_x), \frac{\partial \mathcal{L}}{\partial y^i}(a_x)\right) \in (\mathbf{A}^\#)_x,$$

so that

$$\widehat{\mathcal{L}}(x,y) = \left(x, \sum_{i=1}^{m-1} y^i \frac{\partial \mathcal{L}}{\partial y^i}(x,y) - \mathcal{L}(x,y), \frac{\partial \mathcal{L}}{\partial y^i}(x,y)\right).$$

Since $(x,y) \mapsto (x, \frac{\partial \mathcal{L}}{\partial y^i}(x,y))$ is the Legendre map $\lambda_{\mathcal{L}}$, the composition $\mathcal{H} = \widehat{\mathcal{L}} \circ \lambda^{-1}_{\mathcal{L}}$ is a section of $\mathsf{AV}(\mathbf{A}^\#)$, so

$$\mathcal{H}(x,\xi) = \left(x, \sum_{i=1}^{m-1} y^i(\xi)\xi_i - \mathcal{L}(x, y(\xi)), \xi\right)$$

and we end up with the standard Legendre transform. ■

8. Examples

Example 8.1. For an AV-bundle $\mathbf{Z} = (Z, v_{\mathbf{Z}})$ take as the Lagrangian bundle the AV-bundle $\mathsf{AV}(\mathbf{A})$ over $\mathsf{T}M$ with the special affine (this time, in fact, vector) bundle $\mathbf{A} = \widetilde{\mathsf{T}}\mathbf{Z}$. Such situation we encounter in the analytical mechanics of a relativistic charged particle [26] and in the homogeneous formulation of Newtonian analytical mechanics. We have here $\mathbf{A}^\# = \mathsf{P}\mathbf{Z} \times \mathbb{R}$, $\mathsf{P}\mathbf{A}^\# = \mathsf{T}^*\mathsf{P}\mathbf{Z}$, and

$$\mathcal{E} \colon \mathsf{P}\mathbf{A} = \mathsf{P}\widetilde{\mathsf{T}}\mathbf{Z} \to \mathsf{T}\mathsf{P}\mathbf{Z}.$$

is the canonical isomorphism [31]. Since a choice of a section of $\mathbf{Z}$ gives a 'linearization' $\mathbf{Z} \simeq M \times \mathbb{R}$, so that $\mathsf{P}\mathbf{Z} \simeq \mathsf{T}^*M$ and $\mathsf{P}\widetilde{\mathsf{T}}\mathbf{Z} \simeq \mathsf{T}^*\mathsf{T}M$, we get the canonical Tulczyjew isomorphism $\mathcal{E} : \mathsf{T}^*\mathsf{T}M \to \mathsf{T}\mathsf{T}^*M$ and, in local coordinates, the classical Euler–Lagrange equation. The point here is that we use the correct geometrical object $\widetilde{\mathsf{T}}\mathbf{Z}$ which does not refer to any *ad hoc* choice of a section of $\mathbf{Z}$.

Example 8.2. Let $\mu : M \to \mathbb{R}$ be space–time fibred over time with local coordinates (x^i, t), let $A \subset \mathsf{T}M$ be the Lie affgebroid of those vectors which project onto ∂_t, $\mathsf{V} = \mathsf{V}(A)$ — the bundle of vertical vectors in $\mathsf{T}M$, and let $\mathbf{A} = (A \times \mathbb{R}, (0, 1))$ be the special Lie affgebroid with extended Lie affgebroid structure so that the section $(0, 1)$ is central. In this case $\underline{\mathbf{A}} = A$, $\mathsf{P}\mathbf{A} = \mathsf{T}^*A$, $\mathbf{A}^\# = A^\dagger = \mathsf{T}^*M$, and $\underline{\mathbf{A}^\#} = \mathsf{T}^*M/\langle \mathrm{d}t\rangle = \mathsf{V}^*$. If additionally the space–time is trivial, $M = Q \times \mathbb{R}$, there are identifications: $A = \mathsf{T}Q \times \mathbb{R}$, $\underline{\mathbf{A}^\#} = T^*Q \times \mathbb{R}$, so the affine Tulczyjew triple reads

$$
\begin{tikzcd}[column sep=small]
& \mathsf{T}^*(\mathsf{T}^*Q\times\mathbb{R}) \arrow[rr, "\widetilde{\Pi}_{\mathcal{E}}"] \arrow[ddl] \arrow[dr] & & \mathsf{T}(\mathsf{T}Q\times\mathbb{R}) \arrow[ddl] \arrow[dr] & & \mathsf{T}^*(\mathsf{T}Q\times\mathbb{R}) \arrow[ll, "\mathcal{E}"'] \arrow[ddl] \arrow[dr] & \\
& & \mathsf{T}Q\times\mathbb{R} \arrow[rr, "\rho_{\mathcal{E}}"] \arrow[ddl, "\underline{\eta}"'] & & \mathsf{T}Q\times\mathbb{R} \arrow[ddl] & & \mathsf{T}Q\times\mathbb{R} \arrow[ll, "\rho_{\mathcal{E}}"'] \arrow[ddl] \\
\mathsf{T}^*Q\times\mathbb{R} \arrow[rr, "id"] \arrow[dr] & & \mathsf{T}^*Q\times\mathbb{R} \arrow[dr] & & \mathsf{T}^*Q\times\mathbb{R} \arrow[ll, "id"'] \arrow[dr] & & \\
& Q\times\mathbb{R} \arrow[rr, "id"] & & Q\times\mathbb{R} & & Q\times\mathbb{R} \arrow[ll, "id"'] &
\end{tikzcd}
\quad .
$$

In the standard coordinates, $c^k_{ij} = 0$, $\rho_{\mathcal{E}} = id$, and the Euler–Lagrange equation for a time-dependent Lagrangian $\mathcal{L} : \mathsf{T}Q \times \mathbb{R} \to \mathbb{R}$ can be written as a second-order equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^j}\right)(x, \dot{x}, t) = \frac{\partial \mathcal{L}}{\partial x^j}(x, \dot{x}, t).$$

Example 8.3. The Newtonian space–time N is a system (N, τ, g), where N is a four-dimensional affine space with the model vector space V, together with the time projection $\tau : V \to \mathbb{R}$ represented by a non-zero element of V^*, and an Euclidean metric on $V_0 = \tau^{-1}(0) \subset V$ represented by a linear isomorphism $g\colon V_0 \to V_0^*$.

It is known that the standard framework for analytical mechanics is not appropriate for Newtonian analytical mechanics. It is useful for the frame-dependent formulation of the dynamics only. Because of the Newtonian relativity principle which states that the physics is the same for all inertial observers, the velocity, the momentum, and the kinetic energy have no vector interpretation, as for example the sum of velocities depends strongly on the observer (frame). The equivalence of inertial frames means that all the above concepts are affine in their nature and that they become vectors only after fixing an inertial frame. Of course, to get explicit equations for the dynamics we usually fix a frame, but a correct geometrical model should be frame-independent.

In [6] (see also [4, Example 11]) an affine framework for a frame-independent formulation of the dynamics has been proposed. The presented there construction leads to a special affine space $\mathbf{A}_1$, for which $\underline{\mathbf{A}_1} = V_1 = \tau^{-1}(1)$, and which is equipped with an affine metric, i.e. a mapping $h\colon \underline{\mathbf{A}_1} \to \underline{\mathbf{A}_1^\#}$, with the linear part equal to mg, where m is the mass of the particle. The Lagrangian bundle is $\mathbf{A} = N \times \mathbf{A}_1$. The kinetic part of a Lagrangian is a unique, up to a constant, section ℓ of $\mathsf{AV}(\mathbf{A})$ such that the Legendre mapping ℓ_{eg} equals h. Let $P_1 = \underline{\mathbf{A}_1^\#}$. We have the following, obvious identifications:

- $\mathsf{P}\mathbf{A} = N \times V_1 \times V^* \times P_1 \simeq N \times P_1 \times V^* \times V_1 = \mathsf{P}\mathbf{A}^\#$,
- $\mathsf{T}\underline{\mathbf{A}^\#} = N \times P_1 \times V \times V_0^*$.

With this identifications, the mappings $\mathcal{E}\colon \mathsf{P}\mathbf{A} \to \mathsf{T}\underline{\mathbf{A}^\#}$ and $\widetilde{\Pi}\colon \mathsf{P}\mathbf{A}^\# \to \mathsf{T}\underline{\mathbf{A}^\#}$ read

$$\mathcal{E}\colon \mathsf{P}\mathbf{A} = N \times V_1 \times V^* \times P_1 \ni (x, v, a, p) \mapsto (x, p, v, \overline{a}) \in N \times P_1 \times V \times V_0^* = \mathsf{T}\underline{\mathbf{A}^\#},$$
$$\widetilde{\Pi}\colon \mathsf{P}\mathbf{A}^\# = N \times P_1 \times V^* \times V_1 \ni (x, p, a, v) \mapsto (x, p, v, -\overline{a}) \in \mathsf{T}\underline{\mathbf{A}^\#},$$

where $\overline{a}$ is the image of a with respect to the canonical projection $V^* \to V_0^*$. Now, let us consider a Lagrangian of the form

$$\mathcal{L}\colon N \times E_1 \to A\colon (x, v) \mapsto (x, \ell(v) - \varphi(x)),$$

where φ is a (time-dependent) potential. We have then

$$\mathbf{d}\mathcal{L}\colon N \times E_1 \to \mathsf{P}\mathbf{A}\colon (x, v) \mapsto (x, v, -\mathrm{d}\varphi(x), h(v)) \in \mathsf{P}\mathbf{A}$$

and

$$\mathsf{T}(\mathcal{E} \circ \mathbf{d}\mathcal{L})\colon (x, v, v', v'') \mapsto (x, h(v), v, -\overline{\mathrm{d}\varphi(x)}, v', mg(v''), v'', -\mathsf{T}\overline{\mathrm{d}\varphi}(x, v')).$$

It follows that

$$E_{\mathcal{L}} = \{(x, v, v', v'')\colon v = v', -\overline{\mathrm{d}\varphi(x)} = mg(v'')\},$$

i.e. the Euler–Lagrange equations read

$$\tau(\dot{x}) = 1, \quad \ddot{x} = -\frac{1}{m}\nabla\varphi(x),$$

where $\nabla\varphi(x) = g^{-1}(\overline{\mathrm{d}\varphi(x)})$ is the spatial gradient of φ at $x \in N$.

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